On The Binary Quadratic Equation $x^2 - 18xy + y^2 + 32x = 0$

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Abstract – The binary quadratic equation $x^2 - 18xy + y^2 + 32x = 0$ represents a hyperbola. In this paper we obtain a sequence of its integral solutions and present a few interesting relations among them.

Index Terms – Binary quadratic, Hyperbola, Integral solutions, Parabola, Pell equation.

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1. INTRODUCTION

The binary quadratic Diophantine equations (both homogeneous and non-homogeneous) are rich in variety [1-5]. In [6-12] , the binary quadratic non-homogeneous equations representing hyperbolas respectively are studied for their non-zero integral solutions. These results have motivated us to search for infinitely many non-zero integral solutions of another interesting binary quadratic equation given by $x^2 - 18xy + y^2 + 32x = 0$. The recurrence relations satisfied by the solutions x and y are given. Also a few interesting properties among the solutions are exhibited.

2. METHOD OF ANALYSIS

The Diophantine equation representing the binary quadratic equation to be solved for its non-zero distinct integral solution is

$$x^2 - 18xy + y^2 + 32x = 0 \qquad (1)$$

Substituting the linear transformations

$$x = u + v \quad , \quad y = u - v \qquad (2)$$

in (1) , we have

$$5v^2 - 4u^2 + 8(u + v) = 0 \qquad (3)$$

By performing some simplifications , we obtain

$$V^2 = 20U^2 - 4 \qquad (4)$$

where $V = 5v + 4 \ and \ U = u - 1 \qquad (5)$

The least positive integer solution of (4) is

$$U_0 = 1, V_0 = 4$$

Now , to find the other solutions of (4) , consider the pellian equation

$$V^2 = 20U^2 + 1 \qquad (6)$$

whose fundamental solution is

$$\left(\tilde{U}_0, \tilde{V}_0\right) = (2,9)$$

The other solutions of (6) can be derived from the relations

$$\tilde{V}_n = \frac{f_n}{2}$$

$$\tilde{U}_n = \frac{g_n}{2\sqrt{20}}$$

where

$$f_n = (9 + 2\sqrt{20})^{n+1} + (9 - 2\sqrt{20})^{n+1}$$

$$g_n = (9 + 2\sqrt{20})^{n+1} - (9 - 2\sqrt{20})^{n+1}$$

Applying Brahmagupta lemma between $\left(U_0, V_0\right)$ and $\left(\tilde{U}_n, \tilde{V}_n\right)$, the other solutions of (4) can be obtained from the relation

$$U_{n+1}=\frac{f_n}{2}+\frac{g_n}{\sqrt{5}}$$
$$V_{n+1}=2f_n+\frac{5g_n}{\sqrt{5}} \quad (7)$$

By substituting equation (7) in (5) and using (2) , the non-zero distinct integer solutions of (1) are obtained as follows

$$x_{n+1}=\frac{1}{10}\left[9f_n+4\sqrt{5}g_n\right]$$
$$y_{n+1}=\frac{1}{10}\left[f_n+18\right],\ n=-1,1,3,...$$

The recurrence relations for x_{n+1}, y_{n+1} are respectively

$$x_{n+5}-322x_{n+3}+x_{n+1}=-64$$
$$y_{n+5}-322y_{n+3}+y_{n+1}=-576$$

Some numerical examples of x and y satisfying (1) are given in the table 1 below:

Table-1: Numerical Examples

n	x_{n+1}	y_{n+1}
-1	2	2
1	578	34
3	186050	10370
5	59907458	3338530

From the above table , we observe some interesting relations among the solutions which are presented below:

1. x_{n+1} *and* y_{n+1} both are even
2. Relations among the solutions:
   - ❖ $72x_{n+5}+4608=23184x_{n+3}-72x_{n+1}$
   - ❖ $x_{n+3}+32=5796y_{n+3}-x_{n+1}$
   - ❖ $323x_{n+1}-32=18y_{n+1}+x_{n+3}$
   - ❖ $18x_{n+1}+576=323y_{n+3}-y_{n+5}$
   - ❖ $18y_{n+1}+32=323x_{n+1}-x_{n+3}$
   - ❖ $x_{n+5}+10368=5796y_{n+3}-323x_{n+1}$
   - ❖ $323x_{n+5}+20704=104005x_{n+3}-18y_{n+1}$
   - ❖ $18x_{n+3}=y_{n+3}+y_{n+5}$
   - ❖ $18y_{n+3}-32=x_{n+1}+x_{n+3}$
   - ❖ $y_{n+1}=18x_{n+1}-y_{n+3}$
   - ❖ $323y_{n+3}-576=18x_{n+3}-y_{n+1}$
   - ❖ $18x_{n+5}+576=323y_{n+5}-y_{n+3}$
   - ❖ $18y_{n+5}+32=323x_{n+3}-x_{n+1}$
   - ❖ $y_{n+5}+576=323y_{n+3}-18x_{n+1}$
   - ❖ $323y_{n+5}+576=5796x_{n+3}-y_{n+1}$
   - ❖ $y_{n+1}+576=322y_{n+3}-y_{n+5}$
   - ❖ $\frac{1}{18}\begin{pmatrix}3230x_{3n+3}-10x_{3n+5}\\ -644\end{pmatrix}+3\left[(9+2\sqrt{20})^{n+1}+(9-2\sqrt{20})^{n+1}\right]$ is a cubical integer.
3. Each of the following expressions represents a nasty number:
   - ❖ $\frac{1}{3}\left[3230x_{2n+2}-10x_{2n+4}-608\right]$
   - ❖ $\frac{1}{966}\left[1040050x_{2n+2}-10x_{2n+6}-196416\right]$
   - ❖ $\frac{6}{323}\left[57960x_{2n+2}-10y_{2n+6}-10928\right]$
   - ❖ $1080x_{2n+2}-60y_{2n+4}-96$
   - ❖ $\frac{1}{12}\left[4160200x_{2n+4}-1290x_{2n+6}-829312\right]$
   - ❖ $60y_{2n+2}-96$
   - ❖ $\frac{3}{2}\left[1290y_{2n+4}-720x_{2n+4}-23120\right]$

- $\frac{3}{646}\left[4160200y_{2n+4}-720x_{2n+6}-7490800\right]$
- $\frac{3}{2}\left[231840x_{2n+4}-12920y_{2n+6}-23104\right]$
- $19320y_{2n+4}-60y_{2n+6}-34656$
- $\frac{3}{2}\left[231840x_{2n+6}-4160200y_{2n+6}+7442000\right]$

3. OBSERVATIONS

I. Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of hyperbolas which are presented in the Table 2 below.

Table-2: Hyperbolas

S.No:	*Hyperbola*	(X_n, Y_n)
1.	$80Y_n^2-81X_n^2=103680$	$(3210x_{n+1}-10x_{n+3}-640, 3230x_{n+1}-10x_{n+3}-644)$
2.	$2073680Y_n^2-2099601X_n^2=2.786496385*10^{14}$	$\begin{pmatrix}1033610x_{n+1}-10x_{n+5}-206720,\\ 1040050x_{n+1}-10x_{n+5}-208008\end{pmatrix}$
3.	$80Y_n^2-X_n^2=320$	$(1610x_{n+1}-90y_{n+3}-160, 180x_{n+1}-10y_{n+3}-18)$
4.	$8346320Y_n^2-104329X_n^2=3.483052877*10^{12}$	$\begin{pmatrix}518410x_{n+1}-90y_{n+5}-103520,\\ 57960x_{n+1}-10y_{n+5}-11574\end{pmatrix}$
5.	$5Y_n^2-X_n^2=103680$	$\begin{pmatrix}9302490x_{n+3}-28890x_{n+5}-1854720,\\ 4160200x_{n+3}-1290x_{n+5}-829456\end{pmatrix}$
6.	$8346320Y_n^2-X_n^2=33385280$	$(10x_{n+3}-28890y_{n+1}+52000, 10y_{n+1}-18)$
7.	$5Y_n^2-X_n^2=320$	$\begin{pmatrix}1610x_{n+3}-28890y_{n+3}+51680,\\ 720x_{n+3}-12920y_{n+3}+23112\end{pmatrix}$ $\begin{pmatrix}518410x_{n+3}-28890y_{n+5}-51680,\\ 231840x_{n+3}-12920y_{n+5}-23112\end{pmatrix}$
8.	$8.65363202*10^{11}Y_n^2-X_n^2=3.461452808*10^{12}$	$(10x_{n+5}-9302490y_{n+1}+16744480, 10y_{n+1}-18)$
9.	$5Y_n^2-X_n^2=33385280$	$\begin{pmatrix}1610x_{n+5}-9302490y_{n+3}+16744160,\\ 720x_{n+5}-4160200y_{n+3}+7488216\end{pmatrix}$
10.	$Y_n^2-X_n^2=64$	$\begin{pmatrix}518410x_{n+5}-9302490y_{n+5}+16640800,\\ 231840x_{n+5}-4160200y_{n+5}+7441992\end{pmatrix}$

11.	$25920Y_n^2 - X_n^2 = 103680$	$(10y_{n+3} - 1610y_{n+1} + 2880, 10y_{n+1} - 18)$ $(518410y_{n+3} - 1610y_{n+5} - 930240, 3220y_{n+3} - 10y_{n+5} - 5778)$
12.	$2687489280Y_n^2 - X_n^2 = 1.074995712*10^{10}$	$(10y_{n+5} - 518410y_{n+1} + 933120, 10y_{n+1} - 18)$

II. Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of parabolas which are presented in the Table 3 below.

Table-3: Parabolas

S.No:	*Parabola*	(X_n, Y_n)
1.	$90X_n^2 = 160Y_n - 11520$	$(3210x_{n+1} - 10x_{n+3} - 640, 3230x_{2n+2} - 10x_{2n+4} - 608)$
2.	$1449X_n^2 = 8294720Y_n - 1.923047885*10^{11}$	$\begin{pmatrix}1033610x_{n+1} - 10x_{n+5} - 206720, \\ 1040050x_{2n+2} - 10x_{2n+6} - 196416\end{pmatrix}$
3.	$X_n^2 = 80Y_n - 320$	$(1610x_{n+1} - 90y_{n+3} - 160, 180x_{2n+2} - 10y_{2n+4} - 16)$
4.	$X_n^2 = 25840Y_n - 33385280$	$\begin{pmatrix}518410x_{n+1} - 90y_{n+5} - 103520, \\ 57960x_{2n+2} - 10y_{2n+6} - 10928\end{pmatrix}$
5.	$X_n^2 = 360Y_n - 103680$	$\begin{pmatrix}9302490x_{n+3} - 28890x_{n+5} - 1854720, \\ 4160200x_{2n+4} - 1290x_{2n+6} - 829312\end{pmatrix}$
6.	$X_n^2 = 8346320Y_n - 33385280$	$(10x_{n+3} - 28890y_{n+1} + 52000, 10y_{2n+2} - 16)$
7.	$X_n^2 = -20Y_n - 320$	$\begin{pmatrix}1610x_{n+3} - 28890y_{n+3} + 51680, \\ 720x_{2n+4} - 12920y_{2n+4} + 23120\end{pmatrix}$
8.	$X_n^2 = 20Y_n - 320$	$\begin{pmatrix}518410x_{n+3} - 28890y_{n+5} - 51680, \\ 231840x_{2n+4} - 12920y_{2n+6} - 23104\end{pmatrix}$
9.	$X_n^2 = 8.65363202*10^{11}Y_n - 3.461452808*10^{12}$	$(10x_{n+5} - 9302490y_{n+1} + 16744480, 10y_{2n+2} - 16)$
10.	$X_n^2 = -6460Y_n - 33385280$	$\begin{pmatrix}1610x_{n+5} - 9302490y_{n+3} + 16744160, \\ 720x_{2n+6} - 4160200y_{2n+4} + 7490800\end{pmatrix}$

11.	$X_n^2 = 4Y_n - 64$	$\begin{pmatrix} 518410x_{n+5} - 9302490y_{n+5} + 16640800, \\ 231840x_{2n+6} - 4160200y_{2n+6} + 7442000 \end{pmatrix}$
12.	$X_n^2 = 25920Y_n - 103680$	$(10y_{n+3} - 1610y_{n+1} + 2880, 10y_{2n+2} - 16)$ $\begin{pmatrix} 518410y_{n+3} - 1610y_{n+5} - 930240, \\ 3220y_{2n+4} - 10y_{2n+6} - 5776 \end{pmatrix}$
13.	$X_n^2 = 2687489280Y_n - 1.074995712 * 10^{10}$	$(10y_{n+5} - 518410y_{n+1} + 933120, 10y_{2n+2} - 16)$

4. REMARK

One may also solve (1) by treating it as a quadratic in y. In this case , the corresponding solutions of (1) are

$$x_n = \frac{1}{10}\left[f_n + 2\right]$$

$$y_n = \frac{1}{10}\left[9f_n + 4\sqrt{5}g_n + 18\right],\ n = 0,2,4,...$$

5. CONCLUSION

In this paper, we have made an attempt to obtain a complete set of non-trivial distinct solutions for the non-homogeneous binary quadratic equation. To conclude, one may search for other choices of solutions to the considered binary equation and further, quadratic equations with multi-variables.

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